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2008 J. Phys. A: Math. Theor. 41 085207

(http://iopscience.iop.org/1751-8121/41/8/085207)

J. Phys. A: Math. Theor. **41** (2008) 085207 (9pp) doi:10.1088/1751-8113/41/8/085207

On the spectrum of the Laplace operator of metric graphs attached at a vertex—spectral determinant approach

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Received 1 June 2007, in final form 20 December 2007
Published 12 February 2008
Online at stacks.iop.org/JPhysA/41/085207

Abstract
We consider a metric graph $\mathcal{G}$ made of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ attached at one point. We derive a formula relating the spectral determinant of the Laplace operator $S_{\mathcal{G}}(\gamma) = \det(\gamma - \Delta)$ in terms of the spectral determinants of the two subgraphs. The result is generalized to describe the attachment of n graphs. The formulae are also valid for the spectral determinant of the Schrödinger operator $\det(\gamma - \Delta + V(x))$.

PACS numbers: 02.70.Hm, 02.10.Ox

(Some figures in this article are in colour only in the electronic version)

Introduction

Let us consider a bounded compact domain $\mathcal{D}_1$, part of a manifold. We denote by $\mathrm{Spec}(-\Delta; \mathcal{D}_1)$ the set of solutions E of $-\Delta\psi(r) = E\psi(r)$ with $\psi(r)$ satisfying given boundary conditions at the boundary $\partial\mathcal{D}_1$ (Sturm–Liouville problem). Similarly we consider a second bounded compact domain $\mathcal{D}_2$, distinct from $\mathcal{D}_1$ and denote $\mathrm{Spec}(-\Delta; \mathcal{D}_2)$ the spectrum of the Laplace operator in $\mathcal{D}_2$. Now, if we can glue $\mathcal{D}_1$ and $\mathcal{D}_2$ by identification of parts of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ in order to form a unique compact domain $\mathcal{D}$, the question is: can we relate the spectrum $\mathrm{Spec}(-\Delta; \mathcal{D})$ to $\mathrm{Spec}(-\Delta; \mathcal{D}_1)$ and $\mathrm{Spec}(-\Delta; \mathcal{D}_2)$? The aim of this paper is to discuss this question in the particular case of metric graphs when two graphs are attached at one point. For that purpose the spectral information is encoded in the spectral determinant of the graph $\mathcal{G}$, formally defined as $S_{\mathcal{G}}(\gamma) = \det(\gamma - \Delta)$. We first define basic notations and briefly recall some results on the spectral determinant of metric graphs. We derive the relation between the spectral determinant of a graph in terms of the two subgraph determinants, when subgraphs are attached by one point, as represented on figure 2(c). The relation is generalized

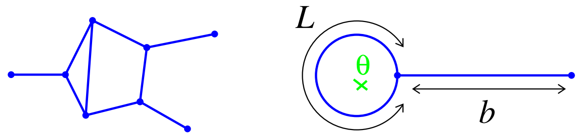

Figure 1. Examples of graphs, (left) a graph with $V = 8$ vertices and $B = 9$ bonds; (right) a ring pierced by a magnetic flux θ attached to a wire ($B = V = 2$).

to describe attachment of $n > 2$ graphs (figure 2(*d*)) and to deal with Schrödinger operator. It is interesting to point out that our result is reminiscent of the gluing formula for elliptic operators acting on a manifold obtained in [1].

Metric graphs

Let us consider a collection of V vertices, denoted here by Greek letters $\alpha, \beta, \ldots$, connected between each others by B bonds, denoted $(\alpha\beta), (\mu\nu), \ldots$. Each bond is associated with two oriented bonds, that we call arcs and denote as $\alpha\beta, \beta\alpha, \mu\nu, \nu\mu, \ldots$. The topology of the graph is characterized by its adjacency (or connectivity) matrix $a_{\alpha\beta}$: $a_{\alpha\beta} = 1$ if $(\alpha\beta)$ is a bond and $a_{\alpha\beta} = 0$ otherwise. The coordination number of the vertex α is denoted, $m_\alpha = \sum_\beta a_{\alpha\beta}$. Up to now we have built a 'combinatorial graph'. If each bond is now identified with an interval $[0, l_{\alpha\beta}] \in \mathbb{R}$, where $l_{\alpha\beta}$ is the length of the bond $(\alpha\beta)$, the set of all connected bonds forms a 'metric graph' (also called a 'quantum graph').

A scalar function $\varphi(x)$ living on a graph $\mathcal{G}$ is defined by B components $\varphi_{\alpha\beta}(x_{\alpha\beta})$, where $x_{\alpha\beta} \in [0, l_{\alpha\beta}]$ is the coordinate along the bond ($x_{\alpha\beta} = 0$ corresponds to vertex α and $x_{\alpha\beta} = l_{\alpha\beta}$ to vertex β). By construction $x_{\alpha\beta} + x_{\beta\alpha} = l_{\alpha\beta}$. Note that components are labelled by arc variables, since we must specify the orientation of the axis along which the coordinate is given; obviously $\varphi_{\alpha\beta}(x_{\alpha\beta}) = \varphi_{\beta\alpha}(x_{\beta\alpha})$ for a scalar function. The action of the Laplace operator on the scalar function along a bond coincides with the one-dimensional Laplace operator $(\Delta\varphi)_{\alpha\beta}(x) = \varphi''_{\alpha\beta}(x)$. In order to define a self-adjoint operator, one must specify boundary conditions at the vertices. The most general conditions have been discussed in [2] (in general the question of boundary conditions is related to the precise nature of the scattering at the vertex [3–6]). In this paper, we consider the simple case of the Laplace operator acting on scalar functions that are continuous at the vertices: $\varphi_{\alpha\beta}(x_{\alpha\beta} = 0) = \varphi_\alpha\ \forall\,\beta$ neighbour of α (that gives $m_\alpha - 1$ equations at the vertex α of coordination number m_α). Then one must impose another condition on derivatives of the function. For continuous boundary condition, the most general condition that ensures self-adjointness of Laplace operator is $\sum_\beta a_{\alpha\beta}\varphi'_{\alpha\beta}(x_{\alpha\beta} = 0) = \lambda_\alpha\varphi_\alpha$ with $\lambda_\alpha \in \mathbb{R}$. The presence of the adjacency matrix in the sum constraints this latter to run over vertices neighbour of α only. The m_α equations ensure self-adjointness of Laplace operator. $\lambda_\alpha = \infty$ corresponds to Dirichlet boundary condition ($\varphi_\alpha = 0$). The study of the Laplace operator on a metric graph appears in several contexts, reviewed in [7, 8], like quantum mechanical problems: $-\Delta\varphi(x) = E\varphi(x)$ could be the Schrödinger equation. In such a case it can be more interesting to consider the situation of a graph submitted to a magnetic field, what is achieved by replacing the derivative by the covariant derivative, $\frac{\mathrm{d}}{\mathrm{d}x} \to \mathrm{D}_x = \frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}A(x)$, where $A(x)$ is the vector potential. The boundary condition then reads $\sum_\beta a_{\alpha\beta}(\mathrm{D}_x\varphi)_{\alpha\beta}(0) = \lambda_\alpha\varphi_\alpha$.

Spectral determinant

The spectral determinant of the Laplace operator Δ is formally defined as $S_{\mathcal{G}}(\gamma) = \det(\gamma - \Delta)$, where γ is the spectral parameter. This object has been introduced in [9] in order to study magnetization of networks of metallic wires. Despite the Laplace operator acts in a space of infinite dimension, $S_{\mathcal{G}}(\gamma)$ can be related to the determinant of a finite size matrix [9],

$$S_{\mathcal{G}}(\gamma) = \prod_{(\alpha\beta)} \frac{\sinh\sqrt{\gamma}l_{\alpha\beta}}{\sqrt{\gamma}} \det\mathcal{M}. \tag{1}$$

Note that the first product, running over all bonds, coincides with the Dirichlet determinant (spectral determinant for Dirichlet conditions at all vertices). A similar decomposition was obtained for combinatorial graphs in [10]. The $V \times V$-matrix $\mathcal{M}$ has matrix elements

$$\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta}\left(\lambda_\alpha + \sqrt{\gamma}\sum_\mu a_{\alpha\mu}\coth\sqrt{\gamma}l_{\alpha\mu}\right) - a_{\alpha\beta}\frac{\sqrt{\gamma}\,\mathrm{e}^{-\mathrm{i}\theta_{\alpha\beta}}}{\sinh\sqrt{\gamma}l_{\alpha\beta}}. \tag{2}$$

This expression describes the case with magnetic field: $\theta_{\alpha\beta}$ is the circulation of the vector potential along the wire $\theta_{\alpha\beta} = \int_{\alpha\beta} \mathrm{d}x\, A(x)$. Generalization to the case of the spectral determinant of the Schrödinger (Hill) operator $S_{\mathcal{G}}(\gamma) = \det(\gamma - \Delta + V(x))$ with generalized boundary conditions has been obtained in [11, 12] (see also the review papers [7, 8]). The result (1) has been derived by two methods: (i) construction and integration of the Kernel of the operator $(-\Delta+\gamma)^{-1}$ [7, 9, 11, 12]: $\int_{\mathcal{G}} \mathrm{d}x \langle x|\frac{1}{-\Delta+\gamma}|x\rangle = \partial_\gamma \ln S_{\mathcal{G}}(\gamma)$. (ii) A path integral derivation [7]. These derivations give the spectral determinant, up to a numerical factor independent on γ (this is inessential for physical quantities since they are always related to $\partial_\gamma \ln S_{\mathcal{G}}$). In the present paper, the precise prefactor of the spectral determinant is fixed by equation (1). Doing so we do not provide a way to determine the γ-independent prefactor from the spectrum of the graph.

It is worth mentioning that the derivation of a ζ-regularized determinant allows us to define the prefactor of the spectral determinant within the calculation. If we denote $\{E_n\}$ the spectrum of an operator $\mathcal{O}$, the determinant of this latter is defined thanks to the ζ-function $\zeta(s) = \sum_n E_n^{-s}$ as $\det_\zeta \mathcal{O} = \exp -\zeta'(0)$ [13]. This approach has been used in [14] where result of [11] for continuous boundary conditions has been obtained with a procedure fixing precisely the multiplicative factor[1].

Attachment of two graphs

Let us consider two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ characterized by matrices $\mathcal{M}_1^{\lambda_\alpha}$ and $\mathcal{M}_2^{\lambda_\beta}$ for generalized boundary conditions at vertices α and β, characterized by parameters λ_α and λ_β. We denote by $S_1^{\lambda_\alpha}(\gamma)$ and $S_2^{\lambda_\beta}(\gamma)$ the corresponding spectral determinants.

[1] We connect notations of [14] with ours. Matrix $A \to$ parameters λ_α's; $\det(R(\lambda) + A) \to \det\mathcal{M}$; the Dirichlet determinant is $\det(H_D + \lambda) \to \prod_{(\alpha\beta)} \frac{2\sinh\sqrt{\gamma}l_{\alpha\beta}}{\sqrt{\gamma}}$. Therefore equation (1.1) of [14] for the ζ-regularized spectral determinant shows that this latter is related to equation (1) by $S_{\mathcal{G}}^{\zeta}(\gamma) = \frac{2^B}{\prod_\alpha m_\alpha} S_{\mathcal{G}}(\gamma)$.

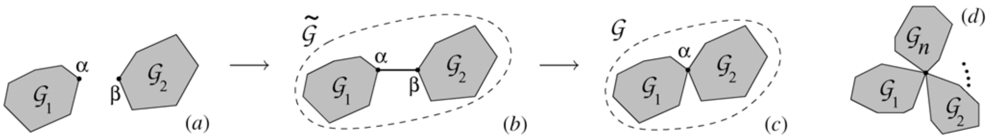

Figure 2. (*a*)–(*c*) Attachment of two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ (the dashed areas hide the structures of the graphs): a bond $(\alpha\beta)$ is introduced, then he limit $l_{\alpha\beta} \to 0$ is taken.

We now attach $\mathcal{G}_1$ and $\mathcal{G}_2$ with a bond $(\alpha\beta)$ (figure 2(*b*)). The new graph is denoted $\widetilde{\mathcal{G}}$. The matrix $\mathcal{M}$ characterizing the new graph has the structure

$$\mathcal{M} = \left(\begin{array}{ccc|ccc} & & & \vdots & & \iddots \\ & \mathcal{M}_1^{\lambda'_\alpha} & & 0 & 0 & \\ & & & -\frac{\sqrt{\gamma}}{\sinh\sqrt{\gamma}l_{\alpha\beta}} & 0 & \cdots \\ \hline \cdots & 0 & -\frac{\sqrt{\gamma}}{\sinh\sqrt{\gamma}l_{\alpha\beta}} & & & \\ & 0 & 0 & & \mathcal{M}_2^{\lambda'_\beta} & \\ \iddots & & \vdots & & & \end{array}\right). \tag{3}$$

The diagonal blocks coincide with the matrices $\mathcal{M}_1^{\lambda_\alpha}$ and $\mathcal{M}_2^{\lambda_\beta}$ of the isolated graphs, provided a modification of the parameters describing boundary condition: $\lambda'_\alpha = \lambda_\alpha + \sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta}$ and $\lambda'_\beta = \lambda_\beta + \sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta}$ (the coth accounts for the additional wire). Then we see that

$$\det\mathcal{M} = \det\left[\mathcal{M}_1^{\lambda'_\alpha}\right]\det\left[\mathcal{M}_2^{\lambda'_\beta} - \mathcal{J}^{(\beta)}\frac{\gamma}{\sinh^2\sqrt{\gamma}l_{\alpha\beta}}\left[\left(\mathcal{M}_1^{\lambda'_\alpha}\right)^{-1}\right]_{\alpha\alpha}\right], \tag{4}$$

where $\mathcal{J}^{(\beta)}$ is the matrix with only one nonzero matrix element equal to 1 on the diagonal corresponding to vertex β: $\mathcal{J}^{(\beta)}_{\mu\nu} = \delta_{\mu\beta}\delta_{\nu\beta}$. Below we introduce the notation $\mathcal{M}_1 \equiv \mathcal{M}_1^{\lambda'_\alpha=\lambda_\alpha}$ and $\mathcal{M}_2 \equiv \mathcal{M}_2^{\lambda'_\beta=\lambda_\beta}$ that denote matrices characterizing $\mathcal{G}_1$ and $\mathcal{G}_2$ before connection. Using

$$\left(\left[\mathcal{M}_1^{\lambda'_\alpha}\right]^{-1}\right)_{\alpha\alpha} = ([\mathcal{M}_1 + \mathcal{J}^{(\alpha)}\sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta}]^{-1})_{\alpha\alpha} = \frac{\left(\mathcal{M}_1^{-1}\right)_{\alpha\alpha}}{1 + \left(\mathcal{M}_1^{-1}\right)_{\alpha\alpha}\sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta}} \tag{5}$$

a little bit of algebra gives

$$\det\mathcal{M} = \det\mathcal{M}_1\det\mathcal{M}_2\left\{1 + \sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta}\left[\left(\mathcal{M}_1^{-1}\right)_{\alpha\alpha} + \left(\mathcal{M}_2^{-1}\right)_{\beta\beta}\right] + \gamma\left(\mathcal{M}_1^{-1}\right)_{\alpha\alpha}\left(\mathcal{M}_2^{-1}\right)_{\beta\beta}\right\}. \tag{6}$$

In the limit $\lambda_\alpha \to \infty$, corresponding to Dirichlet boundary condition, the determinant behaves linearly with λ_α, therefore we define the spectral determinant with Dirichlet boundary condition at vertex α and β as

$$S_1^{\mathrm{Dir}}(\gamma) = \lim_{\lambda_\alpha\to\infty}\frac{S_1^{\lambda_\alpha}(\gamma)}{\lambda_\alpha}, \qquad S_2^{\mathrm{Dir}}(\gamma) = \lim_{\lambda_\beta\to\infty}\frac{S_2^{\lambda_\beta}(\gamma)}{\lambda_\beta}. \tag{7}$$

The Dirichlet determinant is computed by eliminating in $\det\mathcal{M}_1$ the column and the line corresponding to vertex α. Therefore $\left(\mathcal{M}_1^{-1}\right)_{\alpha\alpha} = S_1^{\mathrm{Dir}}/S_1$. Finally, using equations (1) and (6), we obtain for the spectral determinant of the graph of figure 2(*b*),

$$\sqrt{\gamma}S_{\widetilde{\mathcal{G}}}(\gamma) = \cosh\sqrt{\gamma}l_{\alpha\beta}\left[S_1(\gamma)\sqrt{\gamma}S_2^{\mathrm{Dir}}(\gamma) + \sqrt{\gamma}S_1^{\mathrm{Dir}}(\gamma)S_2(\gamma)\right] + \sinh\sqrt{\gamma}l_{\alpha\beta}\left[S_1(\gamma)S_2(\gamma) + \sqrt{\gamma}S_1^{\mathrm{Dir}}(\gamma)\sqrt{\gamma}S_2^{\mathrm{Dir}}(\gamma)\right]. \tag{8}$$

At this stage, it is interesting to discuss the simple case of a graph with Dirichlet boundary at vertices α and β. We can take the limit $\lambda_\alpha, \lambda_\beta \to \infty$, which corresponds to the substitution $S_1 \to \lambda_\alpha S_1^{\mathrm{Dir}}$ and $S_2 \to \lambda_\beta S_2^{\mathrm{Dir}}$. We obtain the expected result $\lim_{\lambda_\alpha,\lambda_\beta\to\infty} \frac{S_{\widetilde{\mathcal{G}}}(\gamma)}{\lambda_\alpha\lambda_\beta} = \frac{\sinh\sqrt{\gamma}l_{\alpha\beta}}{\sqrt{\gamma}} S_1^{\mathrm{Dir}}(\gamma) S_2^{\mathrm{Dir}}(\gamma)$ equivalent to $\mathrm{Spec}(-\Delta; \widetilde{\mathcal{G}}) = \mathrm{Spec}(-\Delta; \mathcal{G}_1) \cup \mathrm{Spec}(-\Delta; \mathcal{G}_2) \cup \left\{\left(\frac{n\pi}{l_{\alpha\beta}}\right)^2; n \in \mathbb{N}^*\right\}$.

The last step of the graph attachment consists to take the limit $l_{\alpha\beta} \to 0$ (figure 2(*c*)) we obtain

$$S_{\mathcal{G}}(\gamma) = S_1(\gamma) S_2^{\mathrm{Dir}}(\gamma) + S_1^{\mathrm{Dir}}(\gamma) S_2(\gamma), \tag{9}$$

which is the central result of the present paper.

Example: Ring attached to a wire (figure 1)

If we consider for $\mathcal{G}_1$ a ring of perimeter L pierced by a flux θ (corresponding to $A(x) = \theta/L$), we have $S_{\mathrm{ring}} = 2(\cosh\sqrt{\gamma}L - \cos\theta)$ and $S_{\mathrm{ring}}^{\mathrm{Dir}} = \frac{\sinh\sqrt{\gamma}L}{\sqrt{\gamma}}$. The graph $\mathcal{G}_2$ is a wire of length b with Neumann boundary at its ends (for a vertex α of coordination number $m_\alpha = 1$ the case $\lambda_\alpha = 0$ coincides with Neumann boundary condition): $S_{\mathrm{wire}}^{\mathrm{both\ Neu}} = \sqrt{\gamma}\sinh\sqrt{\gamma}b$ and $S_{\mathrm{wire}}^{\mathrm{Neu/Dir}} = \cosh\sqrt{\gamma}b$. Therefore, we recover the simple result: $S(\gamma) = \sinh\sqrt{\gamma}b \sinh\sqrt{\gamma}L + 2\cosh\sqrt{\gamma}b(\cosh\sqrt{\gamma}L - \cos\theta)$ obtained directly from (1) in [15].

Equation (8) is contained in equation (9)

The result (9) has appeared as a limit of equation (8), therefore it seems at first sight a particular case of this latter equation. We show now that equation (8) can in fact be recovered from (9). For that purpose we proceed in two steps. First we attach a wire of length b to the graph $\mathcal{G}_1$. The graph formed is denoted $\mathcal{G}_{\mathrm{inter}}$ and the corresponding spectral determinants S_{inter} and $S_{\mathrm{inter}}^{\mathrm{Dir}}$, depending on the nature of the boundary condition at the end of the wire. Spectral determinants of the wire for the three different boundary conditions, $S_{\mathrm{wire}}^{\mathrm{both\ Neu}}$, $S_{\mathrm{wire}}^{\mathrm{Neu/Dir}}$ and $S_{\mathrm{wire}}^{\mathrm{both\ Dir}} \equiv S_{\mathrm{ring}}^{\mathrm{Dir}}$, were given above. Therefore, from equation (9) we obtain

$$S_{\mathrm{inter}} = S_1 S_{\mathrm{wire}}^{\mathrm{Neu/Dir}} + S_1^{\mathrm{Dir}} S_{\mathrm{wire}}^{\mathrm{both\ Neu}} = S_1 \cosh\sqrt{\gamma}b + S_1^{\mathrm{Dir}}\sqrt{\gamma}\sinh\sqrt{\gamma}b \tag{10}$$

$$S_{\mathrm{inter}}^{\mathrm{Dir}} = S_1 S_{\mathrm{wire}}^{\mathrm{both\ Dir}} + S_1^{\mathrm{Dir}} S_{\mathrm{wire}}^{\mathrm{Neu/Dir}} = S_1 \frac{\sinh\sqrt{\gamma}b}{\sqrt{\gamma}} + S_1^{\mathrm{Dir}}\cosh\sqrt{\gamma}b. \tag{11}$$

In a second step we attach the graph $\mathcal{G}_2$ to the end of the wire of $\mathcal{G}_{\mathrm{inter}}$. We use again equation (9) from which it follows that $S_{\widetilde{\mathcal{G}}} = S_{\mathrm{inter}} S_2^{\mathrm{Dir}} + S_{\mathrm{inter}}^{\mathrm{Dir}} S_2$, which precisely coincides with equation (8).

Attachment of two graphs for Schrödinger operator

Let us consider the spectral determinant for the Schrödinger operator (Hill operator) $S_{\mathcal{G}}(\gamma) = \det(\gamma - \Delta + V(x))$ with the same (continuous) boundary conditions as above. Let us first discuss how (1) and (2) are modified. $V_{\alpha\beta}(x_{\alpha\beta})$, with $x_{\alpha\beta} \in [0, l_{\alpha\beta}]$, is the component of the scalar potential $V(x)$ on the bond. An important ingredient is the solution $f_{\alpha\beta}(x_{\alpha\beta})$ of the differential equation $\left[\gamma - \frac{\mathrm{d}^2}{\mathrm{d}x_{\alpha\beta}^2} + V_{\alpha\beta}(x_{\alpha\beta})\right] f_{\alpha\beta}(x_{\alpha\beta}) = 0$ on the interval $[0, l_{\alpha\beta}]$, satisfying $f_{\alpha\beta}(0) = 1$ and $f_{\alpha\beta}(l_{\alpha\beta}) = 0$. A second independent solution of the differential equation

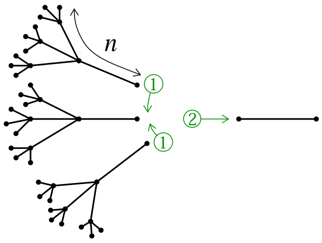

Figure 3. Cayley tree of coordination number z (here 4) and depth n (here 3 before attachment and 4 after).

is $f_{\beta\alpha}(x_{\beta\alpha}) = f_{\beta\alpha}(l_{\alpha\beta} - x_{\alpha\beta})$ (one should not make a confusion: despite using the same notation, the $2B$ functions $f_{\alpha\beta}$ are not the B components of a scalar function). If $V_{\alpha\beta}(x_{\alpha\beta}) = 0$ we have obviously $f_{\alpha\beta}(x) = \frac{\sinh\sqrt{\gamma}(l_{\alpha\beta}-x)}{\sinh\sqrt{\gamma}l_{\alpha\beta}}$. It was shown in [11][2] that equations (1) and (2) are generalized by performing the substitution $\sqrt{\gamma}\coth\sqrt{\gamma}l_{\alpha\beta} \to -f'_{\alpha\beta}(0)$ and $\frac{\sqrt{\gamma}}{\sinh\sqrt{\gamma}l_{\alpha\beta}} \to -f'_{\alpha\beta}(l_{\alpha\beta})$. The matrix $\mathcal{M}$ becomes $\mathcal{M}_{\alpha\beta} = \delta_{\alpha\beta}\left[\lambda_\alpha - \sum_\mu a_{\alpha\mu} f'_{\alpha\mu}(0)\right] + a_{\alpha\beta} f'_{\alpha\beta}(l_{\alpha\beta})\,\mathrm{e}^{-\mathrm{i}\theta_{\alpha\beta}}$ and the spectral determinant takes the form[3] $S_{\mathcal{G}}(\gamma) = \left[\prod_{(\alpha\beta)} f'_{\alpha\beta}(l_{\alpha\beta})\right]^{-1}\det\mathcal{M}$.

We consider two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ on which lives a scalar potential $V(x)$. If these two graphs are attached by a bond $(\alpha\beta)$ where the potential vanishes [$V(x) \neq 0$ for $x \in \mathcal{G}_1 \cup \mathcal{G}_2$ and $V(x) = 0$ for $x \in (\alpha\beta)$], the structure (3) still holds. Therefore all results derived above are still valid, and in particular equations (8), (9) and also (12).

Attachment of *n* graphs

We consider a graph $\mathcal{G}$ obtained by attachment of n graphs at the same point (figure 2(*d*)). It is now easy to generalize (9) in order to describe this situation. We start from (9): $S_{\mathcal{G}} = S_1 S^{\mathrm{Dir}}_{2+\cdots+n} + S^{\mathrm{Dir}}_1 S_{2+\cdots+n}$ and use $S^{\mathrm{Dir}}_{2+\cdots+n} = S^{\mathrm{Dir}}_2\cdots S^{\mathrm{Dir}}_n$. Proceeding by recurrence, we end with

$$S_{\mathcal{G}} = \sum_{k=1}^{n} \underbrace{S^{\mathrm{Dir}}_1\cdots S^{\mathrm{Dir}}_{k-1}}_{\text{Dirichlet}}\, S_k\, \underbrace{S^{\mathrm{Dir}}_{k+1}\cdots S^{\mathrm{Dir}}_n}_{\text{Dirichlet}}. \tag{12}$$

Cayley tree

We can use (12) to study the case of a Cayley tree of coordination number z. We denote S_n the spectral determinant of a Cayley tree of depth n with $\lambda_\alpha = 0\ \forall\,\alpha$. The spectral determinant for the similar graph with Dirichlet boundary at one of its ends is denoted S^{Dir}_n. We proceed in two steps represented on figure 3: first we attach $z-1$ such trees together, using (12). Then

[2] The notations used here are slightly different from those of [11]. They coincide with those of [8, 16].

[3] The fact that $S_{\mathcal{G}}(\gamma) \propto \det\mathcal{M}$ has already been demonstrated in [17]; however, the remaining factor has been obtained in [11] by construction of the resolvent in the graph. Note that the Dirichlet determinant $[\prod_{(\alpha\beta)} f'_{\alpha\beta}(l_{\alpha\beta})]^{-1}$ may play a role in order to determine the full spectrum. A trivial example is the wire (with $V(x) = 0$) for which $\det\mathcal{M} = \gamma$, that does not determine the spectrum. Another example is studied in detail in section 12 of [7]. The γ-dependent factor $[\prod_{(\alpha\beta)} f'_{\alpha\beta}(l_{\alpha\beta})]^{-1}$ is also important from a physical point of view since $\frac{\partial}{\partial\gamma}\ln S(\gamma)$ (for $V(x) = 0$) has been shown to be related to several physical quantities [7–9].

we attach a wire of length b by using (9). We find :

$$S_{n+1} = \left[(z-1)S_n \cosh\sqrt{\gamma}b + \sqrt{\gamma}S_n^{\mathrm{Dir}} \sinh\sqrt{\gamma}b\right]\left(S_n^{\mathrm{Dir}}\right)^{z-2} \tag{13}$$

$$\sqrt{\gamma}S_{n+1}^{\mathrm{Dir}} = \left[(z-1)S_n \sinh\sqrt{\gamma}b + \sqrt{\gamma}S_n^{\mathrm{Dir}} \cosh\sqrt{\gamma}b\right]\left(S_n^{\mathrm{Dir}}\right)^{z-2} \tag{14}$$

with $S_1 = S_{\mathrm{wire}}^{\mathrm{both\,Neu}}$ and $S_1^{\mathrm{Dir}} = S_{\mathrm{wire}}^{\mathrm{Neu/Dir}}$. Note that in the case $z = 2$ the recurrence is trivially solved and give the spectral determinants for a wire of length nb.

Conclusion

Let us come back to the initial question of the paper. Equation (9) seems at first sight to involve only spectral information on graphs $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}$, however we mentioned above that, in equation (1), the γ-independent prefactor is not *a priori* fixed by spectral information. Therefore we can only provide here a partial answer to the initial question: given the spectral determinants of two metric graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, defined by (1), we can determine the spectrum of the graph $\mathcal{G}$ formed by attaching the two graphs at a vertex. An interesting development would be to provide a relation similar to (9) when the spectral determinant and its γ-independent prefactor are constructed from the spectrum only. In the case of ζ-regularization of [14], the relation between ζ-regularized determinant and (1), mentioned in a footnote above, suggests that the relation for ζ-regularized determinants analogous to (9) also involves some information on the coordination numbers of vertices.

The choice of continuous boundary conditions was an important hypothesis in order to derive equations (9), (12). Another simple choice of boundary conditions, assuming continuity of the derivative of the field at the vertices, is examined in the appendix. This leads to a relation with a similar structure, equation (A.2). A question would be to generalize the results (12), (A.2) to the case of general boundary conditions. This would require to formulate the problem with matrices coupling arcs [12] since in the absence of continuity of the field or its derivative, one cannot introduce anymore vertex variables.

An interesting development would be to generalize (9) to other attachment procedures (graphs attached at more than one vertex). For that purpose, a helpful starting point may be the scattering interpretation of equation (9). If a graph is connected to an infinite wire, its spectrum is continuous and we can consider the scattering problem. A plane wave $\mathrm{e}^{-\mathrm{i}kx}$ of energy $E = -\gamma = k^2$ sent from the infinite lead is reflected by the graph with a phase shift $\mathrm{e}^{\mathrm{i}kx+\mathrm{i}\delta(k^2)}$ given by $\mathrm{cotg}[\delta(E)/2] = -\sqrt{E}\frac{S^{\mathrm{Dir}}(-E)}{S(-E)}$, as shown in [7] (equation (117)), where $S^{\mathrm{Dir}}(-E)$ corresponds to Dirichlet boundary condition at the vertex where infinite wire is attached. We can associate with the two graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ two such phase shifts $\delta_1(E)$ and $\delta_2(E)$. The spectrum of the graph $\mathcal{G}$ obtained by the attachment of $\mathcal{G}_1$ and $\mathcal{G}_2$ is given by the Bohr–Sommerfeld quantization condition $\delta_1(E_n) + \delta_2(E_n) = 2n\pi$, which rewrites $\mathrm{cotg}[\delta_1/2] + \mathrm{cotg}[\delta_2/2] = 0$. Since the spectral determinant vanishes on the spectrum, $S(-E_n) = 0$, this shows that $S \propto \frac{S_1^{\mathrm{Dir}}}{S_1} + \frac{S_2^{\mathrm{Dir}}}{S_2}$ (note however that this argument misses a factor function of the energy; see the footnote 3). The scattering problem has been studied for graphs with an arbitrary number of contacts (infinite leads); in particular, expressions of the scattering matrix of a graph with L infinite leads is available in [18] (for $V(x) = 0$) and [6] (for $V(x) \neq 0$). It must also be pointed that the question of graph attachment has been studied in [19] and in particular how to construct the scattering matrix of a graph in terms of subgraphs scattering matrices. All these results on scattering theory in graphs might help the construction of the spectral determinant of two graphs attached by $L > 1$ vertices.

Acknowledgment

I thank Y Colin de Verdière and Alain Comtet for interesting discussions.

Appendix. Derivative continuous at the vertices

The boundary conditions discussed in this paper ($\varphi(x)$ continuous and $\sum_\beta a_{\alpha\beta}\varphi'_{\alpha\beta}(0) = \lambda_\alpha\varphi_\alpha$) can be interpreted as the introduction of a δ-potential at the vertex. They are denoted 'δ-coupling' in [17], where 'δ'-coupling' are also introduced. These latter correspond to continuity of the derivative $\varphi'_{\alpha\beta}(0) = \varphi'_\alpha\ \forall\,\beta$ neighbour of α and $\sum_\beta a_{\alpha\beta}\varphi_{\alpha\beta}(0) = \mu_\alpha\varphi'_\alpha$ (the limit $\mu_\alpha \to \infty$ corresponds to Neumann boundary condition $\varphi'_\alpha = 0$). The results of the present paper are easily generalized to the case of δ'-couplings.

A.1. Spectral determinant

The spectral determinant now involves the solution of the differential equation $\left[\gamma - \frac{d^2}{dx_{\alpha\beta}^2} + V_{\alpha\beta}(x_{\alpha\beta})\right]g_{\alpha\beta}(x_{\alpha\beta}) = 0$ on the interval $[0, l_{\alpha\beta}]$, satisfying $g'_{\alpha\beta}(0) = 1$ and $g'_{\alpha\beta}(l_{\alpha\beta}) = 0$. The spectral determinant is given by $S_{\mathcal{G}}(\gamma) = \left[\prod_{(\alpha\beta)} g_{\alpha\beta}(l_{\alpha\beta})\right]^{-1}\det\mathcal{N}$ with $\mathcal{N}_{\alpha\beta} = \delta_{\alpha\beta}\left[\mu_\alpha - \sum_\nu a_{\alpha\nu}g_{\alpha\nu}(0)\right] - a_{\alpha\beta}g_{\alpha\beta}(l_{\alpha\beta})\,e^{-i\theta_{\alpha\beta}}$. Note that $\left[\prod_{(\alpha\beta)} g_{\alpha\beta}(l_{\alpha\beta})\right]^{-1}$ corresponds to the Neumann determinant: disconnected wires with Neumann boundary conditions $\mu_\alpha \to \infty\forall\alpha$). In the absence of a potential, $V(x) = 0$,

$$\mathcal{N}_{\alpha\beta} = \delta_{\alpha\beta}\left(\mu_\alpha + \frac{1}{\sqrt{\gamma}}\sum_\nu a_{\alpha\nu}\coth\sqrt{\gamma}l_{\alpha\nu}\right) + a_{\alpha\beta}\frac{e^{-i\theta_{\alpha\beta}}}{\sqrt{\gamma}\sinh\sqrt{\gamma}l_{\alpha\beta}} \tag{A.1}$$

and $S_{\mathcal{G}}(\gamma) = \left(\prod_{(\alpha\beta)}\sqrt{\gamma}\sinh\sqrt{\gamma}l_{\alpha\beta}\right)\det\mathcal{N}$.

A.2. Graph attachment

We consider n graphs characterized by spectral determinants S_k for $k = 1, \ldots, n$. We introduce the notation $S_k^{\mathrm{Neu}} = \lim_{\mu_\alpha\to\infty}\frac{S_k}{\mu_\alpha}$, where α is the vertex of attachment of the n graphs (figure 2(*d*)). Since spectral determinants for continuous boundary conditions and continuous derivative have similar structures, the results obtained in this paper are easily generalized. In particular the result (12), from which other results have been derived, becomes for δ'-couplings,

$$S_{\mathcal{G}} = \sum_{k=1}^{n}\underbrace{S_1^{\mathrm{Neu}}\cdots S_{k-1}^{\mathrm{Neu}}}_{\text{Neumann}}\,S_k\,\underbrace{S_{k+1}^{\mathrm{Neu}}\cdots S_n^{\mathrm{Neu}}}_{\text{Neumann}}. \tag{A.2}$$

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